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Section 6.2 Math 12 Honours Basics with Graphing Complex Numbers

1. Given that the modulus of a complex number is $r=\sqrt{a^2+b^2}$, can the modulus be negative? Explain:

No, the radius can never be negative as seen: $\sqrt{a^2+b^2} \geq 0$

2. Given that a complex number in polar form is $a+ib=r(\cos\theta+i\sin\theta)$, why is the argument θ only be between $-\pi$ *and* π ? Explain:

convention?

3. If a complex number is in quadrant 3, what can you tell us about the argument? Explain:

$-90^\circ < \theta < -180^\circ$

4. What happens when you multiply a complex number "z" by just the imaginary value "i"? Explain:

Your answer will rotate 180°.

5. Suppose the imaginary component of a complex number "z" is $\operatorname{Im}(z)=4\cos 17^\circ$ and the modulus of "z' is 4, what the value of $\operatorname{Re}(z)=??$

$r=4$

$\operatorname{Im}(z)=\cancel{r}\sin\theta=\cancel{4}\cos 17^\circ$

$\sin\theta=\cos 17^\circ$

$\theta=73^\circ \Rightarrow \operatorname{Re}(z)=r\cos\theta=\boxed{4\cos 73^\circ}$

6. Given that $\operatorname{Re}(z)=\underbrace{4\cos 30^\circ}_{=2\sqrt{3}}$ and $\operatorname{Im}(z)=\underbrace{4\sin 30^\circ}_{=2}$, what is the $\operatorname{Re}(z\times i^3)$ and $\operatorname{Im}(z\times i^3)$

$\operatorname{Re}(z\cdot i^3)=\operatorname{Re}(i^3 2\sqrt{3}+2i^4)=\boxed{2}$

$\operatorname{Im}(z i^3)=\operatorname{Im}(i^3 2\sqrt{3}+2i^4)=\boxed{-2\sqrt{3}}$

7. Are the two complex numbers the same? Explain:

$z_1=(\cos 240^\circ+i\sin 240^\circ)^3$ and $z_2=(\cos(-120^\circ)+i\sin(-120^\circ))^3$

$z_1=e^{i240^\circ\cdot 3}=e^{720i}=e^0=1$

$z_2=e^{-i120\cdot 3}=e^{-i360^\circ}=e^0=1$

Same

8. Given that $z_1=\cos\theta_1+i\sin\theta_1$ and $z_2=\cos\theta_2+i\sin\theta_2$, what is $z_3=z_1\times z_2$ in polar form??

$z_1=e^{i\theta_1}$

$z_2=e^{i\theta_2}$

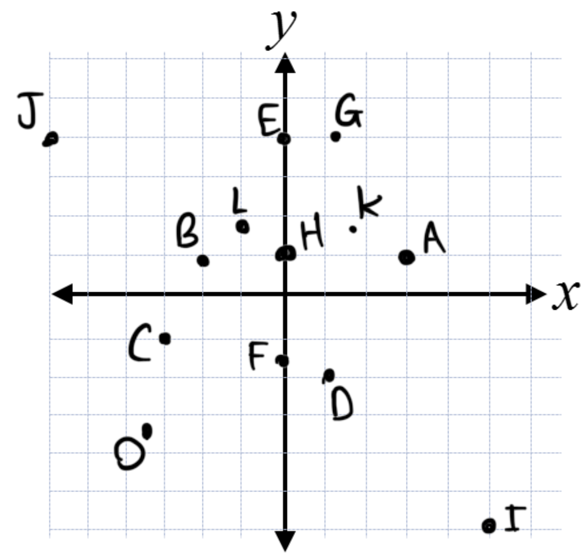
$z_1 z_2=e^{i(\theta_1+\theta_2)}=\boxed{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)}$

9. Given that $z_1=\cos\theta_1+i\sin\theta_1$ and $z_2=\cos\theta_2+i\sin\theta_2$, what is $z_3=z_1\div z_2$ in polar form??

$\frac{e^{i\theta_1}}{e^{i\theta_2}}=e^{i(\theta_1-\theta_2)}=\boxed{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)}$

10. Represent each of the following complex numbers on the Argand Diagram,

A) $3+i$	B) $-2+i$	C) $-3-i$	D) $1-2i$
E) $4i$	F) $-\sqrt{3}i$	G) $\sqrt{2}+4i$	H) i
I) $5-6i$	J) $-6+4i$	K) $2(\cos 45° + i\sin 45°)$ $= \sqrt{2}+\sqrt{2}i$	
L) $-2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$ $-1+\sqrt{3}i$		O) $-5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ $= -\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$	



11. Find the Modulus and Argument for each of the complex numbers and then convert to polar form:

i) $\sqrt{3}+i$ $r=\sqrt{3+1}=2$ $\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=30°$ $z=\boxed{2\cos 30° + 2i\sin 30°}$	ii) $-1-i\sqrt{3}$ $r=\sqrt{1+3}=2$ $\theta=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=-120°$ $z=\boxed{2\cos(-120°)+2i\sin(-120°)}$
iii) $12+5i$ $r=\sqrt{144+25}=13$ $\theta=\tan^{-1}\left(\frac{5}{12}\right)=22.6°$ $z=\boxed{13\cos 22.6° + 13i\sin 22.6°}$	iv) $-\sqrt{2}-\sqrt{2}i$ $r=\sqrt{2+2}=2$ $\theta=\tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right)=-135°$ $z=\boxed{2\cos(-135°)+2\sin(-135°)}$
v) $6i$ $r=6$ $\theta=90°$ $z=\boxed{6\cos 90° + 6i\sin 90°}$	vi) $-4i$ $r=4$ $\theta=-90°$ $z=\boxed{4\cos(-90°)+4\sin(-90°)}$
vii) $3+2i$ $r=\sqrt{9+4}=\sqrt{13}$ $\theta=\tan^{-1}\left(\frac{2}{3}\right)=33.7°$ $z=\boxed{\sqrt{13}\cos 33.7° + \sqrt{13}i\sin 33.7°}$	viii) $-3-3i\sqrt{3}$ $r=\sqrt{9+27}=6$ $\theta=\tan^{-1}\left(\frac{-3\sqrt{3}}{-3}\right)=-120°$ $z=\boxed{6\cos(-120°)+6\sin(-120°)i}$

ix) $(12+3i)(5-4i)$ $60-48i+15i+12$ $=72-33i$ $r=\sqrt{72^2+33^2}=79.2$ $\theta=\tan^{-1}\left(\frac{-33}{72}\right)=-24.6°$ $z=\boxed{79.2\cos(-24.6°)+79.2\,i\sin(-24.6°)}$	x) $-2(\cos 30°+i\sin 30°)\times 3(\underbrace{\cos 45°-i\sin 45°})$ $\cos 45°-i\sin 45°=\cos(-45°)+i\sin(-45°)$ $=-2e^{30°i}\cdot 3e^{-45°i}=\boxed{-6e^{-15°i}}$

12. Reduce the following to rectangular form: $a+ib$

i) $4(\cos 90°+i\sin 90°)$ $=\boxed{4i}$	ii) $3(\cos 240°+i\sin 240°)$ $=\boxed{\frac{3}{2}+\frac{3\sqrt{3}}{2}i}$
iii) $2(\cos 315°+\sin 315°)$ $=2\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=\boxed{\sqrt{2}-\sqrt{2}i}$	iv) $-4(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6})$ $-4\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$ $=\boxed{-2\sqrt{3}+2i}$
v) $-2(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})$ $-2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\boxed{-1+\sqrt{3}i}$	vi) $-5(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$ $=-5\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=\boxed{\frac{5\sqrt{2}}{2}+\frac{5\sqrt{2}i}{2}}$

13. Find $z_1\times z_2$, $\frac{z_1}{z_2}$, and also $\frac{z_2}{z_1}$: $z_1=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ $z_2=\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$z_1=\sqrt{2}e^{\frac{3\pi}{4}i}$ $z_2=\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}i}$

$z_1z_2=\cancel{\sqrt{2}}e^{\frac{3\pi}{4}i}\cdot\frac{1}{\cancel{\sqrt{2}}}e^{\frac{\pi}{4}i}=e^{i\pi}=\boxed{-1}$

$\frac{z_1}{z_2}=\frac{\sqrt{2}e^{\frac{3\pi}{4}i}}{\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}i}}=2e^{\frac{\pi}{2}i}=\boxed{2i}$

$\frac{z_2}{z_1}=\frac{1}{2i}\cdot\frac{xi}{xi}=\boxed{\frac{-i}{2}}$

14. Find $z_1 \times z_2$, $\frac{z_1}{z_2}$, and also $\frac{z_2}{z_1}$: given that : $z_1 = -2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $z_2 = 3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$$Z_1 = -2e^{\frac{2\pi}{3}i} \qquad Z_2 = 3\left[\cos\left(-\tfrac{\pi}{6}\right) + i\sin\left(-\tfrac{\pi}{6}\right)\right] = 3e^{-\frac{\pi}{6}i}$$

$$Z_1 Z_2 = -6e^{\frac{\pi}{2}i} = \boxed{-6i}$$

$$\frac{Z_1}{Z_2} = -\frac{2}{3}e^{\frac{5}{6}\pi} = -\frac{2}{3}\cos 150^\circ - \frac{2}{3}i\sin 150^\circ = \boxed{\frac{\sqrt{3}}{3} - \frac{1}{3}i}$$

$$\frac{Z_2}{Z_1} = \frac{1}{\frac{1}{\sqrt{3}} - \frac{1}{3}i} = \frac{\frac{\sqrt{3}}{3} + \frac{1}{3}i}{\frac{4}{9}} = \boxed{\frac{3\sqrt{3}}{4} + \frac{3}{4}i}$$

15. (1995 AIME) For certain real values a,b,c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13+i$ and the sum of the other two roots is $3+4i$, where $i = \sqrt{-1}$. Find "b"

$r_1 r_2 = 13+i \quad r_3 + r_4 = 3+4i$

Since all coefficients are real, they must appear in conjugate pairs.
since $r_1 r_2$ is imaginary, r_1 and r_2 are NOT conjugate pairs. Neither are r_3 and r_4.

$$b = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4$$

$r_1 = Z_1 = w + xi \quad r_3 = \overline{Z_1} = w - xi$

$r_2 = Z_2 = y + zi \quad r_4 = \overline{Z_2} = y - zi$

$$\boxed{\begin{aligned} w+y &= 3 \\ x+z &= -4 \end{aligned}}$$

$r_1 r_2 = wy + i(wz + xy) - xz = 13 + i$

$$\boxed{wy - xz = 13 \qquad wz + xy = 1}$$

expand "b" and substitute values in for the 4 expressions above.

16. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i$. Find the values of "a", "b", and "c":

$b = 13 + i + w^2 + x^2 + wy - wzi + \cancel{xyi} + xz$
$+ wy - xyi + \cancel{wzi} + xz + y^2 + z^2 + wy$
$- \cancel{wzi} - \cancel{xyi} - xz$

$b = w^2 + 2wy + y^2 + x^2 + 2xz + z^2 + wy - xz$
$+ i(xy + wz) + 13 + i$

$b = (w+y)^2 + (x+z)^2 + (wy - xz) + i(xy + wz) + 13 + i$

$b = 9 + 16 + 13 + \cancel{i} + 13 - \cancel{i} = \boxed{51}$

1

$$\frac{1+\sqrt{3}i}{2} = e^{60^\circ i}$$

$$f(e^{60^\circ i}) = a(e^{60^\circ i})^{2018} + b(e^{60^\circ i})^{2017} + c(e^{60^\circ i})^{2016}$$

$$= ae^{-120^\circ i} + be^{-60^\circ i} + ce^{0^\circ i}$$

$$= a\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + b\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + c$$

$\frac{a\sqrt{3}}{2} + \frac{b\sqrt{3}}{2} = 2019\sqrt{3} \Rightarrow a + b = 4038$ $\boxed{\begin{aligned} a &= 2019 \\ b &= 2019 \end{aligned}}$

Since $a, b \leq 2019$ and $a + b = 4038$, $a = 2019$ and $b = 2019$.
We also know $\frac{b}{2} - \frac{a}{2} + c = 2015$
so $c = 2015$.

$\boxed{(a,b,c) = (2019, 2019, 2015)}$

2

17. (1988 AIME Problem 11) Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n}(z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line.

1

$$\sum_{k=1}^{5}(Z_k - W_k) = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 - 3 - 504i = 0 \Rightarrow Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 3 + 504i$$

$$\sum_{k=1}^{5} \mathrm{Re}(Z_k) = 3 \qquad \sum_{k=1}^{5} \mathrm{Im}(Z_k) = 504$$

let $z_1 = a + bi$
$z_2 = c + di$
$z_3 = e + fi$
$z_4 = g + hi$
$z_5 = p + qi$

2 $(y - y_1) = m(x - x_1)$

$y = mx + (y_1 - mx_1) \Rightarrow y\text{-int} = y_1 - mx_1 = 3$

$$\left.\begin{aligned} b - ma &= 3 \\ c - md &= 3 \\ &\vdots \\ p - mq &= 3 \end{aligned}\right\} \Rightarrow \underbrace{b + d + f + h + q}_{\sum \mathrm{Im}(Z_k) = 504} - m\underbrace{(a + c + e + g + p)}_{\sum \mathrm{Re}(Z_k) = 3} = 15 \Rightarrow 504 - 3m = 15 \Rightarrow \boxed{m = 163}$$